Modified asymmetric explicit group (AEG) FDTD method for TM waves propagation

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***Abstract*—In this paper, a new numerical method for two dimensional electromagnetic wave propagation is presented in order to increase the maximum time step size arising from the stability consideration and to reduce its computational time. The method is based on the modified asymmetric approximation that can easily converted to explicit form. The performance of the method known as modified asymmetric explicit group (AEG) is compared with the conventional finite difference time domain (FDTD) and other methods of natural ordering.**

I. INTRODUCTION

Finite Difference Time Domain (FDTD) method is one of the most commonly used numerical methods for the simulation of wave propagation. This method, known as Yee's algorithm, computes the field components by discretizing the Maxwell's curl equations both in time and space, and then solving the discretized equations in a time marching sequence by alternatively calculating the electric and magnetic fields in the computational domain [1].

Recently, a reduced scalar version of the FDTD method that solves the scalar wave equations was developed by Aoyagi et.al in source free regions [2]. In comparison with the FDTD, the new version which is called the scalar wave equation finite difference time domain (WE-FDTD) method, requires less computation and storage, yet produces similar results as simulated by the conventional FDTD. A major drawback of both the FDTD and the scalar WE-FDTD schemes is that very large computational time and very large computer memory storage are required for analyzing large computational domains. Solving the problems by discretizing its computational domains in a group of points may reduced the computational times and gives as good results as the conventional methods. This was proven when Evans and Abdullah skillfully developed the explicit group method according to the asymmetric Saul'yev scheme and applied the method to the solution of parabolic equations, Burger equations, diffusion equations, etc ([3], [5]). Later Abdullah and Othman [6] developed the techniques known as explicit decoupled method (EDG) and modified explicit group (MEG) method to reduce the algorithm complexity arises by using explicit group method on elliptic problems. All these methods are favorable in parallelism due to their explicit nature.

In this paper, we introduce a new numerical solution based on the explicit group method for a two-space dimensional electromagnetic problem given by the transverse magnetic (TM) waves in free space environment ($E_x = E_y = H_z = 0$)

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_z}{\partial y} \tag{1}$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu_0}\frac{\partial E_z}{\partial x} \tag{2}$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon_0}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] \tag{3}$$

To reduce the algorithm complexity in the TM waves formulations, the equations can be solved simultaneously in source free region as given by the scalar wave-equation

$$\frac{\partial^2 u}{\partial t^2} = c_0^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \tag{4}$$

where c_0 is the speed of light in free space medium and u is either electric or magnetic field component function. In the next section, the formulations of some basic asymmetric finite difference approximations on (4) are presented. Later in section III, we develop a solving formula for a group method of four points using new modified asymmetric scheme. The numerical experiment to test the performance of the method and the results are presented in section IV. Finally, concluding remarks is given in section V.

II. ASYMMETRIC FINITE DIFFERENCE APPROXIMATION

In this section, the numerical solution of (4) on the rectangular solution domain $\Omega = [0, 1] \times [0, 1]$ is considered with initial conditions

$$u(x, y, 0) = \phi(x, y), \qquad u_t(x, y, 0) = \psi(x, y)$$

where c_0 is the speed of light in free space medium and $(x, y) \in \Omega$, $t \in (0, T]$. Introducing $v = \frac{\partial u}{\partial t}$, we can rewrite

(4) as

$$\frac{\partial v}{\partial t} = c_0^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \tag{5}$$

The domain Ω is divided into a uniform grid size $h = \frac{1}{M}$ in both space directions and time increment $k = \frac{T}{N}$ which M and N are both positive integers. Grid points is denoted as $u_{i,j}^n = u(ih, jh, nk)$ and $v_{i,j}^n = v(ih, jh, nk)$ for $i, j = 0, 1, 2, \ldots, M$, $n = 1, 2, \ldots, N$. This equation is in 'almost parabolic' form and can be approximated using different types of asymmetric finite difference formulae.

A simple asymmetric discretization on (5) introduced by Saul'yev [4] using uniform grid space h about a point $(i, j, n+1/2)$ is given as

$$\begin{aligned} -ru_{i+1,j}^{n+1} + 2(1+r)u_{i,j}^{n+1} - ru_{i,j+1}^{n+1} \\ = ru_{i-1,j}^{n} + 2(1-r)u_{i,j}^{n} + ru_{i,j-1}^{n} + 2k\nu_{i,j}^{n} \end{aligned} \tag{6}$$

where $r = (\frac{c_0 k}{h})^2$ represent the courant factor that determine the stability of the method. Another type of asymmetric scheme that can be used to approximate (5) is by rotating the x and y plane axis clockwise by 45^0 with the grid spacing $h \rightarrow \sqrt{2}h$. Thus the rotated asymmetric finite difference approximation for (5) becomes

$$\begin{aligned} -\frac{r}{2}u_{i+1,j+1}^{n+1} + 2(1+\frac{r}{2})u_{i,j}^{n+1} - \frac{r}{2}u_{i-1,j+1}^{nn1} \\ = \frac{r}{2}u_{i-1,j-1}^{n} + 2(1-\frac{r}{2})u_{i,j}^{n} + \frac{r}{2}u_{i+1,j-1}^{n} + 2k\nu_{i,j}^{n} \end{aligned} \tag{7}$$

where the solution vector ν is given by

$$\nu_{i,j}^{n+1} = \frac{2(u_{i,j}^{n+1} - u_{i,j}^{n})}{k} - \nu_{i,j}^{n}$$

III. Modified 4-points Asymmetric Finite Difference Scheme

Consider a group of four points $A(i, j, k), B(i+2, j+2, k), C(i+2, j, k), D(i, j+2, k)$ as illustrated in figure (1) with grid spacing $h \rightarrow 2h$. We approximate (5) at each of these points respectively using basic asymmetric scheme as follows:

$$\begin{aligned} -\frac{r}{4}u_{i+2,j}^{k+1} + au_{i,j}^{k+1} - \frac{r}{4}u_{i,j+2}^{k+1} \\ = \frac{r}{4}u_{i-2,j}^{k} + bu_{i,j}^{k} + \frac{r}{4}u_{i,j-2}^{k} + 2k\nu_{i,j}^{k} \end{aligned} \tag{8}$$

$$\begin{aligned} -\frac{r}{4}u_{i,j}^{k+1} + au_{i+2,j}^{k+1} - \frac{r}{4}u_{i+2,j+2}^{k+1} = \frac{r}{4}u_{i+4,j}^{k} \\ +bu_{i+2,j}^{k} + \frac{r}{4}u_{i+2,j-2}^{k} + 2k\nu_{i+2,j}^{k} \end{aligned} \tag{9}$$

$$\begin{aligned} -\frac{r}{4}u_{i,j+2}^{k+1} + au_{i+2,j+2}^{k+1} - \frac{r}{4}u_{i+2,j}^{k+1} = \frac{r}{4}u_{i+4,j+2}^{k} \\ +bu_{i+2,j+2}^{k} + \frac{r}{4}u_{i+2,j+4}^{k} + 2k\nu_{i+2,j+2}^{k} \end{aligned} \tag{10}$$

$$\begin{aligned} -\frac{r}{4}u_{i,j}^{k+1} + au_{i,j+2}^{k+1} - \frac{r}{4}u_{i+2,j+2}^{k+1} = \frac{r}{4}u_{i-2,j+2}^{k} \\ +bu_{i,j+2}^{k} + \frac{r}{4}u_{i,j+4}^{k} + 2k\nu_{i,j+2}^{k} \end{aligned} \tag{11}$$

with $a = 2(1+\frac{r}{4})$, $b = 2(1-\frac{r}{4})$ and $r = (\frac{c_0 k}{h})^2$ determine the courant stability factor. Therefore, equations (8-11) can be grouped together to give the (4×4) implicit system whose matrix form is given by

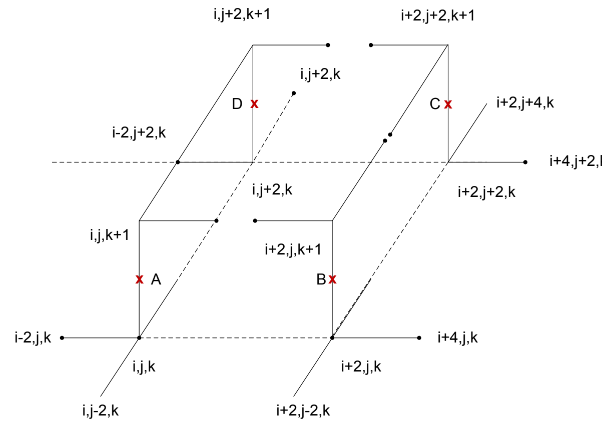

Fig. 1. Group of four points ABCD in two-dimensional space

$$(2I + \frac{r}{4}A)\mathbf{u} = \mathbf{b} \tag{12}$$

where

$$A = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

is a symmetric positive definite matrix and for any group of four-points $\mathbf{u} = (u_{i,j}^{n+1}, u_{i+2,j}^{n+1}, u_{i+2,j+2}^{n+1}, u_{i,j+2}^{n+1})^T$, $\mathbf{b} = (b_1, b_2, b_3, b_4)^T$ where

$$b_1 = \frac{r}{4}u_{i-2,j}^{k} + bu_{i,j}^{k} + \frac{r}{4}u_{i,j-2}^{k} + 2k\nu_{i,j}^{k}$$

$$b_2 = \frac{r}{4}u_{i+4,j}^{k} + bu_{i+2,j}^{k} + \frac{r}{4}u_{i+2,j-2}^{k} + 2k\nu_{i+2,j}^{k}$$

$$b_3 = \frac{r}{4}u_{i+4,j+2}^{k} + bu_{i+2,j+2}^{k} + \frac{r}{4}u_{i+2,j+4}^{k} + 2k\nu_{i+2,j+2}^{k}$$

$$b_4 = \frac{r}{4}u_{i-2,j+2}^{k} + bu_{i,j+2}^{k} + \frac{r}{4}u_{i,j+4}^{k} + 2k\nu_{i,j+2}^{k}$$

From the Lemma of Kellog, the matrix $(2I + \frac{r}{4}A)$ is invertible and

$$\left(2I + \frac{r}{4}A\right)^{-1} = \frac{1}{c} \begin{pmatrix} c_1 & c_2 & c_3 & c_2 \\ c_2 & c_1 & c_2 & c_3 \\ c_3 & c_2 & c_1 & c_2 \\ c_2 & c_3 & c_2 & c_1 \end{pmatrix}$$

where

$$\begin{aligned} c &= \tfrac{r^3}{2} + 5r^2 + 16r + 16 \\ c_1 &= \tfrac{r^3}{16} + \tfrac{r^2}{4} + 6r + 8 \\ c_2 &= \tfrac{r^3}{16} + \tfrac{r^2}{2} \\ c_3 &= \tfrac{r^3}{16} + \tfrac{r^3}{4} \end{aligned}$$

so the equations are written in explicit form as

$$\begin{pmatrix} u_{i,j} \\ u_{i+2,j} \\ u_{i+2,j+2} \\ u_{i,j+2} \end{pmatrix}^{n+1} = \frac{1}{c} \begin{pmatrix} c_1 b_1 + c_2 b_2 + c_3 b_3 + c_2 b_4 \\ c_2 b_1 + c_1 b_2 + c_2 b_3 + c_3 b_4 \\ c_3 b_1 + c_2 b_2 + c_1 b_3 + c_2 b_4 \\ c_2 b_1 + c_3 b_2 + c_2 b_3 + c_1 b_4 \end{pmatrix} \tag{13}$$

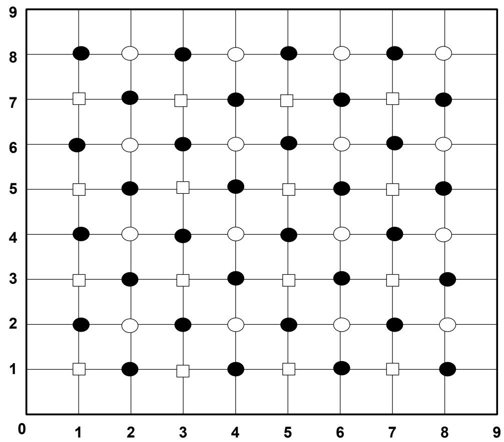

Fig. 2. MEG solution domain with $n = 9$

This gives the solving formula for a modified group of four points at each time level.

Each scheme (8-11) has a local truncation error of approximately $O(\frac{k}{h} + h^2 + k^2)$ with the coefficient to k/h being of opposite signs. Due to the difference in signs of truncation error in each scheme, the alternating error will tend to cancel the effect of k/h term at most internal points, thus leaving the accuracy of to approximately of $O(h^2 + k^2)$. The solving formula (10) is implemented iteratively until a convergence criteria is met. The computational molecule of the modified explicit system can be described in figure (2). From the figure it can be observed that the iterative evaluation of solution (5) only involve points of type $\circ$ until a convergence criteria is met. The remaining points can be evaluated explicitly at the required time steps using direct formula. Therefore the execution time can be saved by nearly a quarter as the iteration only carried out on quarter of the computational domain.

For simplicity, we consider the number of interior points $(m - 1)$ to be an even number such that each group $G_l, l = 1, 2, ...$ consists of four points in natural ordering as described in (2). We can define the modified four points explicit group method (M-4pEG) on (5) as following:

i) Divide the solution domain into grid with odd number of grid line as illustrated in figure (2)
ii) Calculate group of 4-points marked as $\circ$ as following:
   - Iterate solution of the group points $\circ$ using (13)
   - Implement the relaxation procedure (*SOR*)
   - check convergence. If converge stop iteration, otherwise re-initialise and do iteration
iii) Calculate points marked as $\bullet$ explicitly using formula (7) at the required time steps
iv) Calculate points marked as $\diamond$ explicitly using formula (6) at the required time steps

IV. Numerical Experiment

To test the performance of the presented method, we perform the numerical simulation on (4) in a lossless medium with normalized electric permittivity and magnetic permeability, that is $\varepsilon = \mu = 1$. We set the solution region as $\Omega = [0, 1] \times [0, 1]$ surrounded by PEC boundary conditions. The exact solution of the problem is given as follows:

$$u(x, y, t) = \sqrt{2}\cos(\sqrt{2}\pi t)\sin[\pi(1 - x)]\sin[\pi(1 - y)]$$

The experiment was run on a Sun-Fire-v240 machine with one processor running and carried out with different mesh sizes and courant factors (r). The graphs of the simulation are shown in figure (3-4).

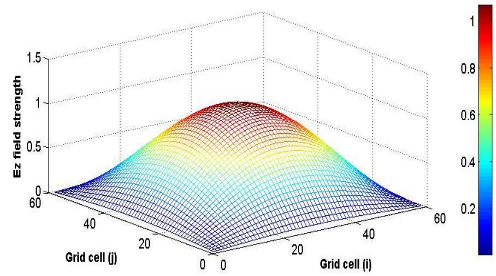

Fig. 3. TM wave propagation after 20 time steps using modified asymmetric explicit group (AEG) method with $r = 0.5$.

TABLE I
COMPARISON OF AVERAGE ABSOLUTE ERROR (A.A.E) OF THE METHODS WITH $h^{-1} = 33$ AFTER 10 TIME STEPS

Courant factor, r	FDTD	4pAEG	M-4pAEG
0.1	3.74e-4	2.15e-5	1.05e-5
0.25	3.50e-3	2.98e-4	9.11e-4
0.5	1.30e-2	9.40e-4	4.88e-3
1.0	4.47e-2	3.26e-3	1.94e-3
2.0	unstable	3.86e-2	8.54e-2

TABLE II
COMPARISON OF AVERAGE ABSOLUTE ERROR (A.A.E) OF THE METHODS WITH $h^{-1} = 65$ AFTER 10 TIME STEPS

Courant factor, r	FDTD	4pAEG	M-4pAEG
0.1	1.40e-4	2.78e-6	5.46e-5
0.25	8.66e-4	3.86e-5	1.25e-5
0.5	3.40e-3	1.26e-4	7.88e-3
1.0	1.28e-2	4.86e-4	1.94e-3
2.0	unstable	3.34e-3	7.21e-3

The performance of the new method compared to available results obtained from the conventional FDTD [1] method and the standard four points EG-FDTD [7] in terms of maximum error (M.E), average absolute error (A.A.E) and CPU elapsed time are shown in table (I-II).

V. Conclusion

In this paper, we present a modified explicit group scheme based on the asymmetric approximation in order to increase

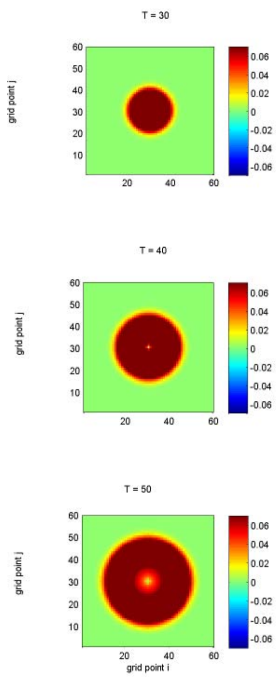

Fig. 4. TM wave propagation after various time steps using modified asymmetric explicit group (AEG) method with $r = 0.5$.

TABLE III
COMPARISON OF CPU ELAPSED TIME (SEC) AFTER 10 TIME STEPS

h^{-1}	FDTD	4pAEG	M-4pAEG
17	0.15	0.78	0.19
25	0.47	1.56	0.39
33	0.63	3.18	0.74
65	1.78	5.28	1.22
81	3.55	14.28	4.33

the maximum time step size arising from the stability consideration and reduce the computational time. The method can be easily converted to explicit form. The results show that the presented modified method produces as good result as the conventional FDTD method. Furthermore the modified scheme can reduced the cpu time to nearly a quarter than the 4pAEG method. Overall it can conclude than the presented method relatively good as one of the alternative to solve problems related to waves propagation.

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